JHEP HYPER VERSION

Received: *January 29, 1999,* Accepted: *February 26, 1999*

Branes at generalized conifolds and toric geometry

Rikard von Unge

Department of Theoretical Physics and Astrophysics
Faculty of Science, Masaryk University
Kotlářská 2, 611 37, Brno, Czech Republic
E-mail: unge@physics.muni.cz

Abstract: We use toric geometry to investigate the recently proposed relation between a set of D3 branes at a generalized conifold singularity and type IIA configurations of D4 branes stretched between a number of relatively rotated NS5 branes. In particular we investigate how various resolutions of the singularity corresponds to moving the NS branes and how Seiberg's duality is realized when two relatively rotated NS-branes are interchanged.

Keywords: D-branes, Brane Dynamics in Gauge Theories.

JHEP02(1999)023

Contents

1. Introduction

The duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and supergravity on the space $AdS_5 \times S^5$ recently conjectured by Maldacena [1] was found by studying the physics and the near horizon geometry of D3-branes in flat space. Therefore, to try to extend this conjecture it is natural to study D3-branes on more complicated spaces. The most natural generalization is perhaps to study D3-branes at an orbifold singularity [2, 3].

In this context, it is also useful to study T-dual versions of these models. Under T-duality the singularity gets mapped into a configuration of NS5-branes and the D3-branes gets mapped into D4-branes. The so called Brane Box [4, 5] models are related by two T-duality transformations to D3-branes at an orbifold singularity $\mathbb{C}^3/\Gamma$. The T-duality transformations maps parameters and moduli of the field theory of the branes into geometrical quantities of the T-dual brane configuration which makes certain phenomena more easily studied. One can for instance realize Seiberg's duality as a "reshuffling" of branes [6] and in some cases solve the field theory by lifting the brane configuration to M-theory [7].

More recently, following the work by Klebanov and Witten [8], D3-branes on non-orbifold singularities, leading to gauge theories which are not just projections of $\mathcal{N} = 4$ theory, have been studied [9, 10, 11, 12, 13, 14, 15, 16, 17]. The basic example of [8] concerned D3-branes on a conifold singularity and the subsequent work is various generalizations of that.

One interesting article is [17] where D3-branes on a quotient of the conifold by an appropriate discrete isometry group were studied. A duality between D3-branes on these singular spaces and configurations of NS-branes and D4-branes in type IIA theory was also proposed and the relation between resolutions of the singularity and movement of the branes was studied. Furthermore it was proposed that Seiberg's duality (the interchanging of NS and NS′ branes in this context) could be realized as flop transitions between topologically different small resolutions of the singularities.

The purpose of this paper is to use toric geometry (for an introduction see [18]) to study this correspondence in more detail and thereby provide additional evidence for the proposed duality and the correspondence between the resolution of the singularity and the movement of the branes. Using toric methods we will be able to show that the Higgs branch of the moduli space of the gauge theory is indeed the generalized conifold singularity. We will also be able to get an explicit handle on the various resolutions of this singularity by including the FI-terms in the calculation.

Toric geometry has been used previously in the study of 3-branes at various singularities. The methods used in this paper were introduced in [3] where D3-branes on $\mathbb{C}^3/Z_k$ orbifolds were studied. In particular it was shown that only geometric phases of the singularity was seen by the D3-branes. More complicated orbifold singularities were studied in [19, 20, 21, 10], including conifiold singularities as subsingularities. It was also shown that the D-branes are sensitive to topologically different resolutions of the singularities, differing by flop transitions.

This paper is organized as follows: section 2 studies the simplest non-trivial example and explicitly derives the toric data describing this singularity. In section 3 we use the toric data to study how various resolutions are related to movement of the NS-branes in the T-dual version of the model. In section 4 we study a more complicated example corresponding to the space $xy = z^3w^3$ and use that to make claims about the general correspondence between resolutions of the singularity and movement of the NS-branes. In section 5 we use our methods to study a situation related to Seiberg's duality as was proposed in [17] and in section 6 we summarize the paper and discuss some directions for future research. The paper is finished with an appendix which contains the toric data for some of the more complicated examples studied in the paper.

2. The basic example

We are interested in studying N D3-branes on a singularity of the type $xy = z^k w^{k'}$. According to [17] this is T-dual to a configuration with N D4-branes wound around a compact direction (x_6 in our case) and stretched between k NS-branes and k' NS′-

branes which are placed at various points along the circle. The branes are oriented as follows:

$$\begin{array}{lcccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ NS & \times & \times & \times & \times & \times & \times & - & - & - & - \\ NS' & \times & \times & \times & \times & - & - & - & - & \times & \times \\ D4 & \times & \times & \times & \times & - & - & \times & - & - & - \end{array} \tag{2.1}$$

where a cross means of infinite extent and a dash means point like in the particular dimension.

Let us study the simplest non-trivial example, namely the configuration with two NS-branes and two NS$'$-branes given in figure 1 and, according to [17], T-dual to D3-branes on a singularity of the type $xy = z^2w^2$.

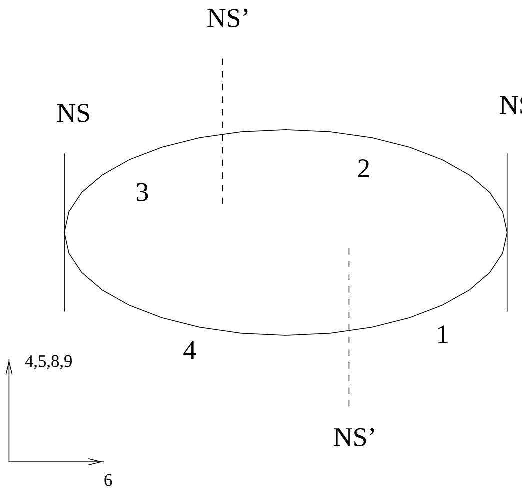


Figure 1: The brane configuration.

The D-term equations for this model are

$$\begin{aligned} |Q_{41}|^2 - \left|\tilde{Q}_{14}\right|^2 - |Q_{12}|^2 + \left|\tilde{Q}_{21}\right|^2 &= \zeta_1\,, \\ |Q_{12}|^2 - \left|\tilde{Q}_{21}\right|^2 - |Q_{23}|^2 + \left|\tilde{Q}_{32}\right|^2 &= \zeta_2\,, \\ |Q_{23}|^2 - \left|\tilde{Q}_{32}\right|^2 - |Q_{34}|^2 + \left|\tilde{Q}_{43}\right|^2 &= \zeta_3\,, \\ |Q_{34}|^2 - \left|\tilde{Q}_{43}\right|^2 - |Q_{41}|^2 + \left|\tilde{Q}_{14}\right|^2 &= \zeta_4\,, \end{aligned} \tag{2.2}$$

and the superpotential is

$$W \propto \tilde{Q}_{21}Q_{12}Q_{23}\tilde{Q}_{32} - \tilde{Q}_{32}Q_{23}Q_{34}\tilde{Q}_{43} + \tilde{Q}_{43}Q_{34}Q_{41}\tilde{Q}_{14} - \tilde{Q}_{14}Q_{41}Q_{12}\tilde{Q}_{21}\,, \tag{2.3}$$

giving the F-term constraints

$$\begin{aligned} Q_{12}\tilde{Q}_{21} &= Q_{34}\tilde{Q}_{43}\,, \\ Q_{23}\tilde{Q}_{32} &= Q_{41}\tilde{Q}_{14}\,. \end{aligned} \tag{2.4}$$

We can solve the F-term constraints in terms of a minimal set of fields. If we choose to have $\tilde{Q}_{43}$ and $\tilde{Q}_{14}$ as dependent on the other fields, the solution can be represented as follows

$$\begin{array}{ccccccc} & Q_{12} & \tilde{Q}_{21} & Q_{23} & \tilde{Q}_{32} & Q_{34} & Q_{41} \\ Q_{12} & 1 & 0 & 0 & 0 & 0 & 0 \\ \tilde{Q}_{21} & 0 & 1 & 0 & 0 & 0 & 0 \\ Q_{23} & 0 & 0 & 1 & 0 & 0 & 0 \\ \tilde{Q}_{32} & 0 & 0 & 0 & 1 & 0 & 0 \\ Q_{34} & 0 & 0 & 0 & 0 & 1 & 0 \\ \tilde{Q}_{43} & 1 & 1 & 0 & 0 & -1 & 0 \\ Q_{41} & 0 & 0 & 0 & 0 & 0 & 1 \\ \tilde{Q}_{14} & 0 & 0 & 1 & 1 & 0 & -1 \end{array}\,. \tag{2.5}$$

To be able to put the F-term constraints and the D-term constraints on equal footing (as a symplectic quotient) we will introduce homogeneous coordinates $p_0 \dots p_7$ as follows

$$
\begin{aligned}
&Q_{12} = p_3 p_4 \;\; Q_{23} = p_2 p_6 \;\; Q_{34} = p_3 p_5 \;\; Q_{41} = p_2 p_7 \\
&\tilde{Q}_{21} = p_1 p_5 \;\; \tilde{Q}_{32} = p_0 p_7 \;\; \tilde{Q}_{43} = p_1 p_4 \;\; \tilde{Q}_{14} = p_0 p_6 \,,
\end{aligned}
\tag{2.6}
$$

in which the F-term constraints are automatically satisfied. The homogeneous coordinates span a $\mathbb{C}^8$ which is acted upon by a $\mathrm{U}(1)^2$ action under which the original Q and $\tilde{Q}$ fields are the invariant coordinates. The solution can also be represented in the form of the matrix

$$
T = \begin{array}{c}
\begin{matrix} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{matrix} \\
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array} ,
\tag{2.7}
$$

where to find which power of p_i a particular Q contains, one takes the corresponding column in (2.7) and take the scalar product with the corresponding row in (2.5).

The F-term constraints can now be represented as a symplectic quotient on the space spanned by the homogeneous coordinates. The symplectic quotient is implemented by the previously mentioned $\mathrm{U}(1)^2$ action under which the homogeneous coordinates has the following charges

$$
\begin{array}{c}
\begin{matrix} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{matrix} \\
\begin{pmatrix}
0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\end{array} .
\tag{2.8}
$$

Now we have to find how the ordinary D-term constraints will act on the homogeneous coordinates. To do this we introduce the matrix U defined by $TU^{tr} = Id$

$$
U = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0
\end{pmatrix} .
\tag{2.9}
$$

Then the D-term constraints are represented by the charge matrix VU where V contains the charges of the basic fields under the particular gauge group. In our case we have three independent gauge groups (the charges of the fourth one is given in terms of the other three) so V is given by

$$V = \begin{pmatrix} Q_{12} & \tilde{Q}_{21} & Q_{23} & \tilde{Q}_{32} & Q_{34} & Q_{41} \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}, \tag{2.10}$$

giving us a charge matrix

$$VU = \begin{pmatrix} 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \end{pmatrix}. \tag{2.11}$$

Now we concatenate the charge matrix from the F-term constraints with the charge matrix representing the D-term constraints to get the full reduction on the $\mathbb{C}^8$ spanned by the homogeneous coordinates. The cokernel of the transpose of this matrix gives us the toric data for the space of interest [3]. In our case it is given by

$$\tilde{T} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{2.12}$$

All of these vectors lie with their tip on the plane with normal $(1,1,1)$ at a distance of $1/\sqrt{3}$ from the origin. With the notation

$$\begin{array}{lll} w_1 = (1,-1,1) & w_2 = (0,-1,2) & w_3 = (1,1,-1) \\ w_4 = (0,1,0) & w_5 = (0,0,1) & w_6 = (1,0,0) \end{array} \tag{2.13}$$

we can draw a picture (figure 2) of where the vectors hit the plane in which we recognize the toric description of the space $xy = z^2w^2$ which is in agreement with the proposal of [17].

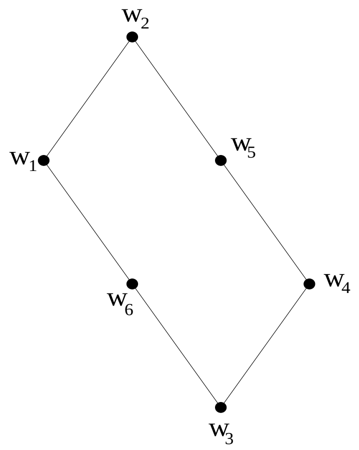


Figure 2: The singularity.

As usual, when we include the FI-terms we resolve the space. Because we have no FI-parameters for the F-term constraints not all possible phases are realized and only geometrical phases are seen exactly as in [3]. To show this and to further study these resolutions we therefore include the FI-parameters and give the charge matrix in a particularly useful form.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \zeta_1+\zeta_2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & \zeta_2+\zeta_3 \\ 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & \zeta_1 \\ 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & \zeta_2 \\ -1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & \zeta_3 \end{pmatrix}. \tag{2.14}$$

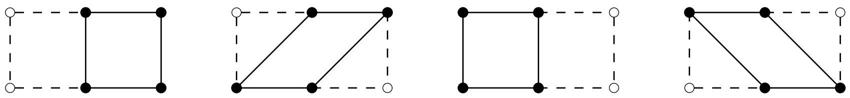
The first two rows can be used to eliminate some of the homogeneous coordinates for different values of $\zeta_1 + \zeta_2$ and $\zeta_2 + \zeta_3$. The result is that only the geometric phases are realized.

For reference we give the data for the different type of singularities here. These can all be obtained from the matrix above by invertible row operations and we have assumed that $\zeta_1 + \zeta_2 \geq 0$ and $\zeta_2 + \zeta_3 \geq 0$. We also give the pictures showing which singularity is controlled by which parameter. The solid lines indicate the singularity and the dotted lines represent the resolved part.

The data for the conifold singularities (of the type $xy = zw$) is

$$\begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & \zeta_1 \\ 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & \zeta_2 \\ -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & \zeta_3 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & \zeta_4 \end{pmatrix}, \tag{2.15}$$

corresponding to the pictures

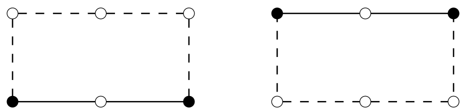


$\zeta_1 = 0$ $\zeta_2 = 0$ $\zeta_3 = 0$ $\zeta_4 = 0$

For the orbifold singularities (of the type $xy = z^2$) it is

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & \zeta_1 + \zeta_2 \\ 0 & 1 & 0 & 1 & 0 & -2 & 0 & 0 & \zeta_2 + \zeta_3 \end{pmatrix}, \tag{2.16}$$

corresponding to the pictures

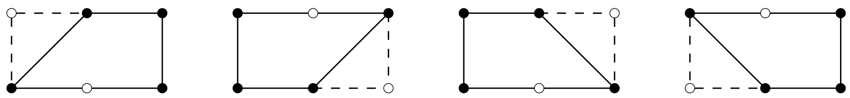


$\zeta_1 + \zeta_2 = 0$ $\zeta_2 + \zeta_3 = 0$

and finally for the suspended pinch point singularities (of the type $xy = zw^2$) it is

$$\begin{pmatrix} -1 & 0 & 1 & -2 & 0 & 2 & 0 & 0 & \zeta_1 - \zeta_2 \\ 2 & -1 & 0 & 1 & 0 & 0 & -2 & 0 & \zeta_2 - \zeta_3 \\ -1 & 2 & 1 & 0 & 0 & -2 & 0 & 0 & \zeta_3 - \zeta_4 \\ 0 & -1 & -2 & 1 & 0 & 0 & 2 & 0 & \zeta_4 - \zeta_1 \end{pmatrix}, \tag{2.17}$$

corresponding to the pictures



$\zeta_1 = \zeta_2$ $\zeta_2 = \zeta_3$ $\zeta_3 = \zeta_4$ $\zeta_4 = \zeta_1$

3. Analysis of subsingularities

Using these expressions we see that for particular values of the FI-parameters we will have various unresolved subsingularities in our configuration. It is interesting to analyze how the brane picture corresponds to the singularity picture for these cases.

If we start with the orbifold singularities we see that if we for example take $\zeta_1 + \zeta_2 = 0$ but leave ζ_3 arbitrary we will leave the first one of them unresolved. Looking at the D-term equations and assuming ζ_1 and ζ_3 are positive we see that they are satisfied if we give expectation values to the chiral fields as follows

$$\begin{aligned}\tilde{Q}_{21} &= \zeta_1^{\frac{1}{2}} ,\\ \tilde{Q}_{43} &= \zeta_3^{\frac{1}{2}} .\end{aligned} \tag{3.1}$$

These expectation values corresponds to moving both the NS-branes in the x_7-direction. This will break the first and the second gauge group down to their diagonal component and similarly for the third and the fourth gauge group. Furthermore, if we insert these expectation values in the superpotential we see that the fields Q_{12} and Q_{34} becomes the adjoint fields of the unbroken gauge groups and we get the usual elliptic $N = 2$ model with two gauge groups [7].

A slightly more complicated example is to study what happens when we leave one of the conifold singularities unresolved. By studying the matrix (2.15) we see that this happens when we for instance put $\zeta_1 = 0$. Inspection of the D-term constraints reveals that they are satisfied if we put

$$\begin{aligned}\tilde{Q}_{32} &= \zeta_2^{\frac{1}{2}} ,\\ \tilde{Q}_{43} &= (\zeta_2 + \zeta_3)^{\frac{1}{2}} ,\end{aligned} \tag{3.2}$$

which corresponds to moving the second NS-brane and the second NS′-brane in the x_7-direction. When we insert this into the superpotential we get

$$\begin{aligned}W \propto\ & \zeta_2^{\frac{1}{2}} \tilde{Q}_{21} Q_{12} Q_{23} - \zeta_2^{\frac{1}{2}} (\zeta_2 + \zeta_3)^{\frac{1}{2}} Q_{23} Q_{34} + \\ & + (\zeta_2 + \zeta_3)^{\frac{1}{2}} Q_{34} Q_{41} \tilde{Q}_{14} - \tilde{Q}_{14} Q_{41} Q_{12} \tilde{Q}_{21} ,\end{aligned} \tag{3.3}$$

This will give a mass to the fields Q_{23} and Q_{34}, and integrating them out we find

$$W \propto Q_{41} \tilde{Q}_{14} \tilde{Q}_{21} Q_{12} - \tilde{Q}_{14} Q_{41} Q_{12} \tilde{Q}_{21} , \tag{3.4}$$

which is indeed the superpotential for the conifold as proposed in [8].

We can also ask what will happen if we leave one of the suspended pinch point singularities unresolved. We see that this will happen if we choose $\zeta_1 = \zeta_2$. However, we are still free to choose $\zeta_1 + \zeta_2$. Let us study the simplest possibility $\zeta_1 + \zeta_2 = 0$

first. This means that $\zeta_1 = \zeta_2 = 0$ and we can assume that $\zeta_3 > 0$. By inspection we find that the D-term constraints are satisfied if we choose

$$\tilde{Q}_{43} = \zeta_3^{\frac{1}{2}}\,, \tag{3.5}$$

which corresponds to moving the second NS-brane in the x_7-direction. Again we break the third and fourth gauge groups to the diagonal combination and Q_{34} becomes a chiral superfield in the adjoint of the unbroken part of that group. The superpotential becomes

$$W \propto \tilde{Q}_{21}Q_{12}Q_{23}\tilde{Q}_{32} - \tilde{Q}_{14}Q_{41}Q_{12}\tilde{Q}_{21} + \zeta_3^{\frac{1}{2}}\left(\tilde{Q}_{14}Q_{34}Q_{41} - Q_{23}Q_{34}\tilde{Q}_{32}\right), \tag{3.6}$$

which is the superpotential one would get from a model with two NS′-branes and only one NS-brane.

If we in the last example choose $\zeta_1 + \zeta_2 > 0$ instead we still keep the suspended pinch point singularity unresolved. However, we will resolve some of its subsingularities giving us a more complicated situation. The D-term equations tells us that we have to give expectation values to three of the hypermultiplets which seems to be a highly nontrivial situation. How come the suspended pinch point singularity is still unresolved? The answer can be found if one uses that we know that the FI-parameter for a particular gauge group is geometrically encoded as the difference in the x_7 coordinate of the NS-brane to the right of of the gauge group and to the left of the gauge group [6]. From this follows that the final configuration still has two NS′-branes and one NS-brane in a line, although this line is slightly tilted in the x_6, x_7 plane.

We may also reverse the question and ask what happens with the singularity when we move a particular NS-brane or NS′-brane in a particular way instead of asking what happens with the brane configuration when we resolve the singularity. Using the correspondence between FI-parameters and the relative x_7 coordinates of the branes again, it is possible to find out exactly what FI-parameters a particular movement corresponds to. For instance, if we move the NS-brane separating the first and the second gauge groups a distance ζ in the negative x_7 direction, we will get the following FI-parameters

$$\begin{aligned} \zeta_1 &= -\zeta\,, \qquad \zeta_2 = \zeta\,, \\ \zeta_3 &= 0\,, \qquad \zeta_4 = 0\,, \end{aligned} \tag{3.7}$$

which, using the D-term equations, corresponds to giving an expectation value to $Q_{12} = \zeta^{\frac{1}{2}}$. Since $\zeta_1 + \zeta_2 \geq 0$ and $\zeta_2 + \zeta_3 \geq 0$ we can use the previous formulas to see that the unresolved singularities are the third suspended pinch point singularity and its subsingularities.

4. More examples

After studying the simplest example in detail we may now turn to investigate more general models such as the one with 3 NS-branes and 3 NS$'$-branes as shown in figure 3.

We can use the method described above to solve for the Higgs branch of the moduli space of the field theory describing this brane configuration. The toric data and the charge matrix giving this moduli space as a symplectic quotient is given in formula (A.2) in the appendix. The toric data corresponds to a singularity of the type $xy = z^3 w^3$ as expected from the arguments given in [17]. We can use the charge matrix to investigate what phases are realized when we put 3-branes on such a singularity and also to find out how moving the NS- and NS$'$-branes are related to various resolutions of the singularity. In particular, there are four different ways of moving the NS- and NS$'$-branes in the x_7 direction (turning on FI-terms). Namely, we can move an NS-brane in the negative x_7 direction, we can move an NS$'$-brane in the negative x_7 direction, we can move an NS-brane in the positive x_7 direction and finally we can move an NS$'$-brane in the positive x_7 direction. By using the charge matrix (A.2) given in the appendix we see that these four distinct ways of moving the branes correspond to four distinct ways of resolving the singularity. These are given in the following figure, in the order corresponding to the movements given in the text:

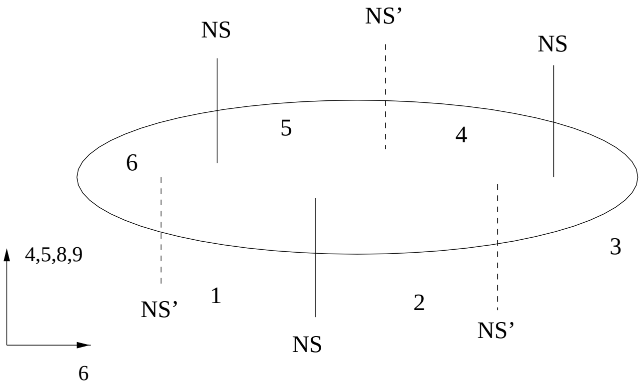
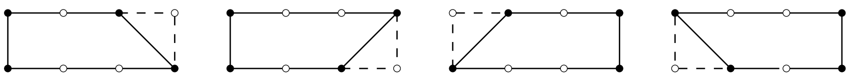


Figure 3: A more complicated example.



It should be pointed out that it does not matter which NS-brane or NS$'$-brane one is moving for this correspondence to hold. The only thing that matters is if it is an NS-brane or an NS$'$-brane or if we move it in the positive or negative x_7 direction. We may also check that this is true for the previously studied configuration given in figure 1.

5. Seiberg's duality

Finally we would like to study what happens to the singularity when one moves an NS-brane past an NS$'$-brane in the x_6 direction. This should be related to Seiberg's duality as was observed in [6]. In [17] this fact was used to conjecture that Seiberg's duality in the "D4-branes between NS-branes" picture is related to flop transitions between topologically distinct small resolutions of the singularity in the T-dual picture. We will now investigate this claim using toric methods.

We can study this phenomenon (following [6]) without encountering the singularity that would result from actually letting the NS-brane and the NS′-brane meet in space by letting one of the branes move in the x_7 direction before moving it in the x_6 direction. Then we are free to move the brane in the x_6 direction until we pass the NS′-brane since they are at different points in x_7 and after the branes have passed each other we can let the branes move back to their original positions in x_7 giving us a configuration where the NS and NS′-brane has changed places.

From the previous discussion we know what happens when we move a brane in the x_7 direction and the singularity structure does not depend on the x_6 coordinates of the NS-branes so what we have to study is what happens when we move the NS-brane in the x_7 coordinate in the configuration where one NS-brane and one NS′-brane has changed places. To be concrete, let us therefore investigate the configuration with two NS-branes and two NS′-branes related to the previously studied configuration by an interchange of the NS-brane and the NS′-brane surrounding the fourth gauge group. This configuration is given in figure 4.

The field theory corresponding to this configuration also has Coulomb branches (corresponding to moving the D4-branes between the NS-branes or between the NS′-branes) but since we are only interested in the Higgs branches we will set the vacuum expectation value of the adjoint chiral superfield which exists for the first and third gauge groups to zero. Doing this and repeating the previous analysis we get the same toric data as in the first example. However, the charge matrix is slightly different and is given in the appendix. In particular, the parameters that control the sizes of the orbifold singularities are ζ_1 and ζ_3.

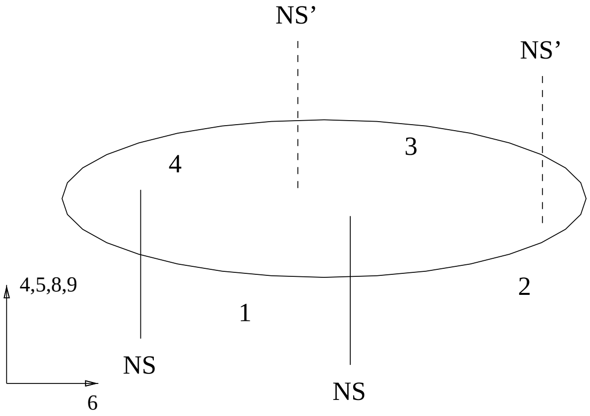


Figure 4: The dual configuration.

If we repeat the analysis performed in section 3 for the configuration in figure 1 we find that the same general results hold. The singularity is resolved in the same way, for instance, if we move any one of the NS-branes in the positive x_7 direction we will remove an upper left triangle from the toric diagram just as for the original configuration. However, the FI-parameters that control the sizes of the various singularities are now different and we may ask if it is possible to map them into each other. To make the question more precise let us study the original configuration where we move the NS-brane on the left of the fourth gauge group a distance ζ in the positive x_7 direction and the NS$'$-brane on the right a distance ϵ in the negative x_7 direction. This corresponds to introducing FI-parameters with the following values

$$
\begin{aligned}
\zeta_1 &= \epsilon\,, & \zeta_2 &= 0\,,\\
\zeta_3 &= \zeta\,, & \zeta_4 &= -\zeta-\epsilon\,,
\end{aligned}
\tag{5.1}
$$

and we see that in particular $\zeta_1+\zeta_2 \geq 0$ and $\zeta_2+\zeta_3 \geq 0$ so we can use the formulas given in the text. On the other hand, in the dual configuration (where this NS-brane and NS$'$-brane have changed places) this would correspond to a set of FI-parameters

$$
\begin{aligned}
\tilde{\zeta}_1 &= -\zeta\,, & \tilde{\zeta}_2 &= 0\,,\\
\tilde{\zeta}_3 &= -\epsilon\,, & \tilde{\zeta}_4 &= \zeta+\epsilon\,,
\end{aligned}
\tag{5.2}
$$

where we see that $\zeta_1 \leq 0$ and $\zeta_3 \leq 0$ which determines which of the homogeneous coordinates to use for the dual configuration.

Now we can compare parameters. For each of the subsingularities there is a particular combination of FI-parameters which controls its size. If we assume that nothing happens with the sizes when we move in x_6 we can equate them and we get this system of equations (one equation for each subsingularity)

$$
\begin{aligned}
\zeta_1+\zeta_2 &= -\tilde{\zeta}_3\,, & \zeta_2+\zeta_3 &= -\tilde{\zeta}_1\,,\\
\zeta_1 &= \tilde{\zeta}_1+\tilde{\zeta}_4\,, & \zeta_2 &= \tilde{\zeta}_2\,,\\
\zeta_3 &= -\tilde{\zeta}_1-\tilde{\zeta}_2\,, & \zeta_4 &= -\tilde{\zeta}_4\,,\\
\zeta_1-\zeta_2 &= -2\tilde{\zeta}_2-\tilde{\zeta}_3, & \zeta_2-\zeta_3 &= \tilde{\zeta}_1+2\tilde{\zeta}_2\,,\\
\zeta_3-\zeta_4 &= \tilde{\zeta}_3+2\tilde{\zeta}_4\,, & \zeta_4-\zeta_1 &= -\tilde{\zeta}_1-2\tilde{\zeta}_4\,,
\end{aligned}
\tag{5.3}
$$

where ζ_i are the FI-parameter in the original model and $\tilde{\zeta}_i$ are the FI-parameter in the dual configuration. This has a solution which is

$$
\begin{aligned}
\tilde{\zeta}_1 &= \zeta_1+\zeta_4\,, & \tilde{\zeta}_2 &= \zeta_2\,,\\
\tilde{\zeta}_3 &= -\zeta_1-\zeta_2\,, & \tilde{\zeta}_4 &= -\zeta_4\,,
\end{aligned}
\tag{5.4}
$$

which means that it seems to be possible to move the NS-branes in the x_6 direction while keeping the sizes of the various singularities fixed. Furthermore, the FI-parameters in (5.4) precisely corresponds to the FI-parameters one would expect if one just interchanged the NS-brane and the NS$'$-brane surrounding the fourth gauge group. To see this we can assume that the x_7 positions of the NS-branes are (with the convention that the NS$'$-brane between the fourth and the first gauge group is the first brane)

$$
\begin{aligned}
x_7(1) &= 0\,, & x_7(2) &= \zeta_1\,,\\
x_7(3) &= \zeta_2+\zeta_1\,, & x_7(4) &= -\zeta_4\,.
\end{aligned}
\tag{5.5}
$$

If we interchange the the first and the fourth NS-branes the positions would instead be

$$\begin{aligned} x_7(\tilde{1}) &= -\zeta_4\,, & x_7(\tilde{2}) &= \zeta_1\,, \\ x_7(\tilde{3}) &= \zeta_2+\zeta_1\,, & x_7(\tilde{4}) &= 0\,, \end{aligned} \tag{5.6}$$

which gives exactly the FI-parameters in (5.4).

We thus see that there is a one to one map between the FI-parameters between the two configuration related by interchanging of one NS-brane and one NS′-brane. In particular, the various subsingularities of the generalized conifold will have the same sizes in both configurations. This means that in this formalism we do not see any sign that the two configuration are related by a flop transition between topologically different small resolutions of the singularities as was conjectured in [17].[1]

6. Conclusions

We have shown how the toric description of the moduli space of D3-branes at a generalized conifold singularity is related to the T-dual version with D4-branes suspended between NS-branes and NS′-branes. By studying various subsingularities of the space it was possible to investigate how moving the NS or NS′ branes in the x_7 direction in the dual configuration resolves the singularity in various ways.

We also showed that with toric methods we were able to study what happens as move two relatively rotated NS-branes across each other in the x_6-direction, corresponding to performing Seiberg's duality in one of the gauge groups of the dual configuration. In [17] it was conjectured that this should be related to flop transitions in the toric diagram. We showed that there is a one to one map between the FI-parameters of the original model and the Seiberg dual model and that this map exactly corresponds to what one would get by naively moving the branes past each other, keeping their position in the x_7 direction. Thus it was not possible to confirm this particular conjecture of [17].[2]

In conclusion one can say that we have given evidence that Toric geometry offers a powerful tool for studying the relation between brane configurations and branes at singularities. It would be interesting to investigate more complicated singularitites, for example of the type $x^k y^k = z^{k'} w^{k'}$, in an analogous way.

[1] The real situation is slightly more complicated however. The x_6 position of the branes are encoded in the integrals over the B-field over the two-cycles of the singularity. When we change the relative x_6 position of the branes we change the B-field on the corresponding cycle and what happens when two branes cross is that the B-field on the cycle in question actually flips sign. What we have shown is that the real size, as measured by the imaginary part of the complexified Kähler form $B + iJ$, does not change when the NS-branes cross. However, since the real part of the complexified Kähler form changes, there might still be something like a flop transition taking place in the complexified Kähler moduli space. It would be interesting to continue this work along this line of reasoning. I would like to thank A. Uranga for discussions on this topic.

[2] See however previous footnote.

Acknowledgments

The author would like to thank Ulf Lindström and Yuri Shirman for useful and stimulating discussions.

A. Additional toric data and charge matrices

Following the procedure outlined in the text we arrive at the following toric data for the Higgs branch of the moduli space of the model given in figure 3.

$$T = \begin{pmatrix} 2 & 2 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & 1 & 2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}. \tag{A.1}$$

It corresponds to the toric data for a space of the type $xy = z^3 w^3$ as expected. The charge matrix for the symplectic quotient (including the FI-parameters) is given by

$$Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2+\zeta_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_6+\zeta_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3+\zeta_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_1+\zeta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & \zeta_4+\zeta_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & \zeta_2+\zeta_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & \zeta_5+\zeta_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \zeta_3+\zeta_4 \end{pmatrix}. \tag{A.2}$$

The toric data for the Higgs branch of the moduli space of the model given in figure 4 is the same as for the model given in figure 1. The D-term equations are also the same. However, the charge matrix implementing the symplectic quotient is different and is given by

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & \zeta_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & \zeta_3 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 & 0 & \zeta_1 \\ 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & \zeta_2 \\ -1 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & \zeta_3 \end{pmatrix}. \tag{A.3}$$

References

[1] J. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] M.R. Douglas and G. Moore, *D-branes, Quivers and ALE Instantons*, hep-th/9603167.

[3] M.R. Douglas, B.R. Greene and D.R. Morrison, *Nucl. Phys.* **B 506** (1997) 84 [hep-th/9704151].

[4] A. Hanany and A. Zaffaroni, *J. High Energy Phys.* **05** (1998) 001 [hep-th/9801134].

[5] A. Hanany and A. Uranga, *J. High Energy Phys.* **05** (1998) 013 [hep-th/9805139].

[6] S. Elitzur, A. Giveon and D. Kutasov, *Phys. Lett.* **B 400** (1997) 269 [hep-th/9702014].

[7] E. Witten, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166].

[8] I.R. Klebanov and E. Witten, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[9] B.S. Acharya, J.M. Figueroa-O'Farrill, C.M. Hull and B. Spence, *Branes at conical singularities and holography*, hep-th/9808014.

[10] D. Morrison and M.R. Plesser, *Non-Spherical Horizons, I*, hep-th/9810201.

[11] K. Oh and R. Tatar, *Three Dimensional SCFT from M2 Branes at Conifold Singularities*, *J. High Energy Phys.* **02** (1999) 025 [hep-th/9810244].

[12] B.R. Greene,C.I. Lazaroiu and M. Raugas, *D-Branes on nonabelian threefold quotient singularities*, hep-th/9811201.

[13] K. Dasgupta and S. Mukhi, *Brane Constructions, Conifolds and M-Theory*, hep-th/9811139.

[14] S.S. Gubser, N. Nekrasov and S. Shatashvili, *Generalized Conifolds and Four Dimensional $\mathcal{N}=1$ Superconformal Theories*, hep-th/9811230.

[15] E. Lopez, *A Family of $\mathcal{N}=1$ $\mathrm{SU}(N)^k$ Theories from Branes at Singularities*, *J. High Energy Phys.* **02** (1999) 019 [hep-th/9812025].

[16] R. de Mello Koch, K. Oh and R. Tatar, *Moduli Space for Conifolds as Intersection of Orthogonal D6 branes*, hep-th/9812097.

[17] A.M. Uranga, *Brane Configurations for Branes at Conifolds*, *J. High Energy Phys.* **01** (1999) 022 [hep-th/9811004].

[18] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Nucl. Phys.* **B 416** (1994) 414 [hep-th/9309097].

[19] T. Muto, *Nucl. Phys.* **B 521** (1998) 183 [hep-th/9711090].

[20] B.R. Greene, *Nucl. Phys.* **B 525** (1998) 284 [hep-th/9711124].

[21] S. Mukhopadhyay and K. Ray, *Phys. Lett.* **B 423** (1998) 247 [hep-th/9711131].